Digital control

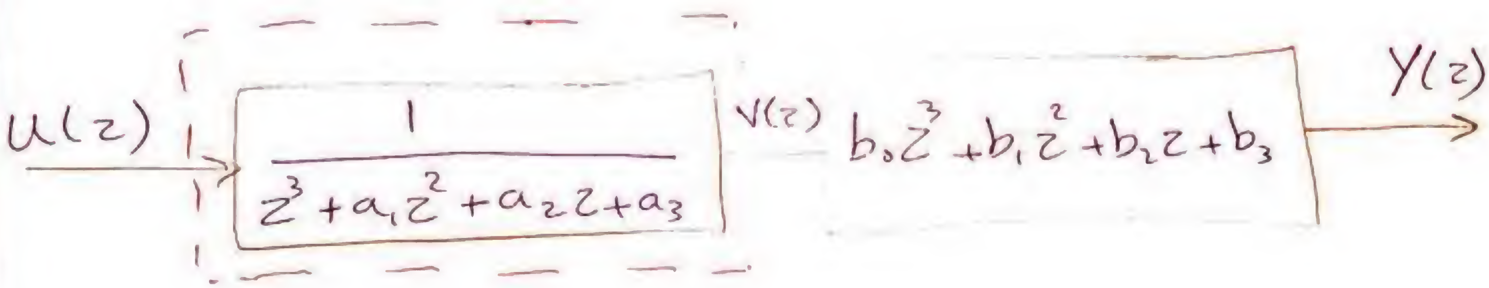
$$T.F = \frac{z^3 + 2z + 3}{z^3 + 4z^2 + 2z - 1}$$

→ Find the state-space model in controllable and observable form.

→ Draw state diagram for each form

*3rd order system

$$T.F = \frac{Y(z)}{U(z)} = \frac{b_0 z^3 + b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}$$



$$\frac{V(z)}{U(z)} = \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3}$$

$$U(z) = (z^3 + a_1 z^2 + a_2 z + a_3)\, V(z) \Big\downarrow z^{-1}.T$$

$$u(K) = V(K+3) + a_1 V(K+2) + a_2 V(K+1) + a_3 V(K) \quad \rightarrow (1)$$

$$x_1(K) = V(K)$$

$$x_2(K) = V(K+1) = x_1(K+1)$$

$$x_3(K) = V(K+2) = x_2(K+1)$$

$$x_3(K+1) = V(K+3)$$

In Continous

$$\dot{x}(t) = A x(t) + B u(t)$$

$$y(t) = C x(t) + D u(t)$$

In discrete

$$x(K+1)T = A x(KT) + B u(KT)$$

$$y(KT) = C x(KT) + D u(KT)$$

from 1

$$V(K+3) = -a_3 V(K) - a_2 V(K+1) - a_1 V(K+2) + u(K)$$

$$\boxed{x_3(K+1) = -a_3 x_1(K) - a_2 x_2(K) - a_1 x_3(K) + u(K)}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

البسط
← ~~المقام~~ هيجيب المعادله التانية .

$$\frac{Y(z)}{V(z)} = b_0 z^3 + b_1 z^2 + b_2 z + b_3$$

$$Y(z) = (b_0 z^3 + b_1 z^2 + b_2 z + b_3)\, V(z) \quad \Big\downarrow z^{-1}.T$$

$$y(k) = b_0 V(k+3) + b_1 V(k+2) + b_2 V(k+1) + b_3 V(k)$$

$$y(k) = b_0\, x_3(k+1) + b_1 x_3(k) + b_2 x_2(k) + b_3 x_1(k)$$

$$= b_0\big(-a_3 x_1(k) - a_2 x_2(k) - a_1 x_3(k) + u(k)\big)$$

$$+ b_1 x_3(k) + b_2 x_2(k) + b_3 x_1(k)$$

$$y(k) = (b_3 - b_0 a_3)\, x_1(k) + (b_2 - a_2)\, x_2(k) +$$

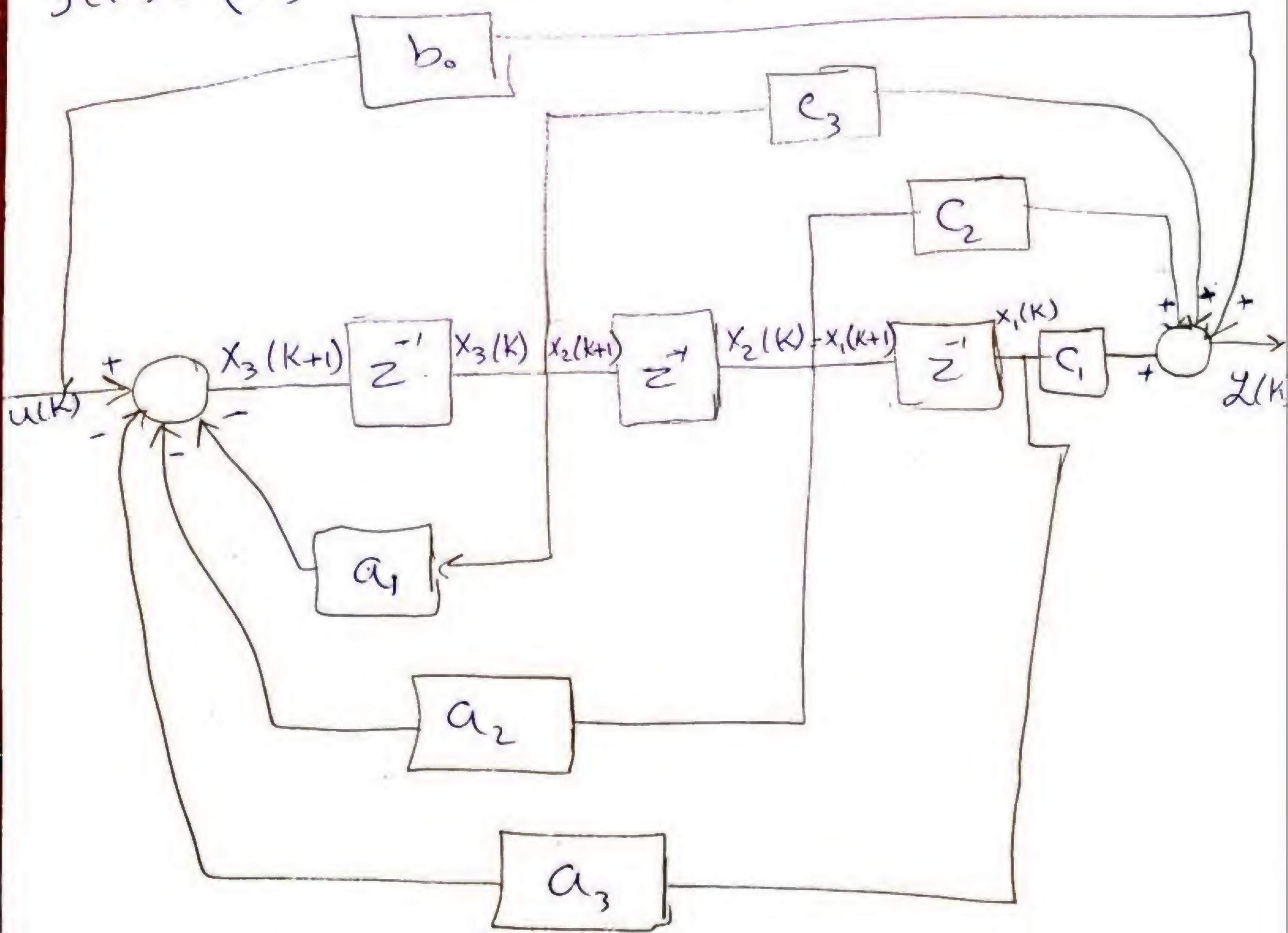
$$(b_1 - a_1)\, x_3(k) + b_0 u(k)$$

$$y(K) = \left(\underbrace{b_3 - b_0 a_3}_{C_1} \quad \underbrace{b_2 - b_0 a_2}_{C_2} \quad \underbrace{b_1 - b_0 a_1}_{C_3} \right) \begin{bmatrix} X_1(K) \\ X_2(K) \\ X_3(K) \end{bmatrix} .$$

$$+ \, b_0 \, u(K)$$

if $b_0 = 0$

$$y(K) = \left(b_3 \quad b_2 \quad b_1 \right) X(K) .$$



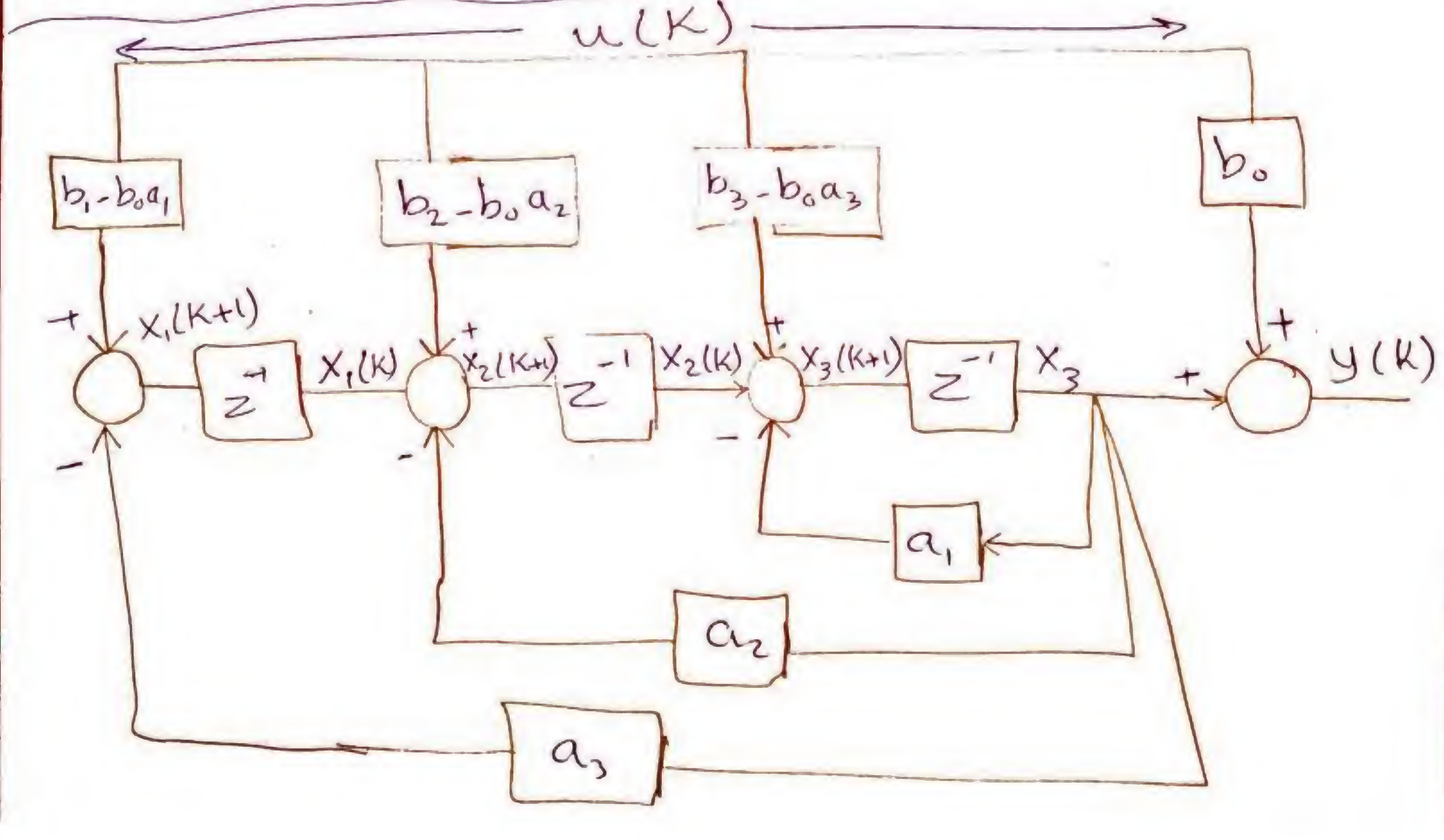
* <u>observable form:</u>

$$A_o = A_c^T \quad ; \quad C_o = B_c^T$$

$$B_o = C_c^T \quad ; \quad D_o = D_c = b_0$$

$$X(K+1) = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix} X(K) + \begin{pmatrix} b_3 - b_0 a_3 \\ b_2 - b_0 a_2 \\ b_1 - b_0 a_1 \end{pmatrix} u(K)$$

$$y(K) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} X(K) + b_0 u(K)$$



$\boxed{\text{EX}}$

$$T.F = \frac{z^3 + 2z + 3}{z^3 + 4z^2 + 2z - 1}$$

Controllable form:-

$$x(K+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -4 \end{pmatrix} x(K) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(K)$$

$$y(K) = \begin{pmatrix} \underbrace{4}_{3-1(-1)} & \underbrace{0}_{2-1(2)} & \underbrace{-4}_{0-1(-4)} \end{pmatrix} x(K) + \underbrace{\cancel{1}}_{b_0 = 1} u(K)$$

observable form

$$x(K+1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -4 \end{pmatrix} x(K) + \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix} u(K)$$

$$y(K) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x(K) + u(K)$$

Diagonal form

Ex

$$T.F = \frac{2z^3 + z + 1}{z(z+0.5)(z-0.5)}$$

$\hookrightarrow z(z^2 - 0.25)$

$$\begin{array}{r|l} & \quad 2 \\ \hline z^3 - 0.25z & \quad 2z^3 + z + 1 \\ & -2z^3 \overset{+}{-} 0.5z \\ \hline & \quad 1.5z + 1 \end{array}$$

$$\boxed{T.F = \frac{1.5z + 1}{z(z+0.5)(z-0.5)} + 2}$$

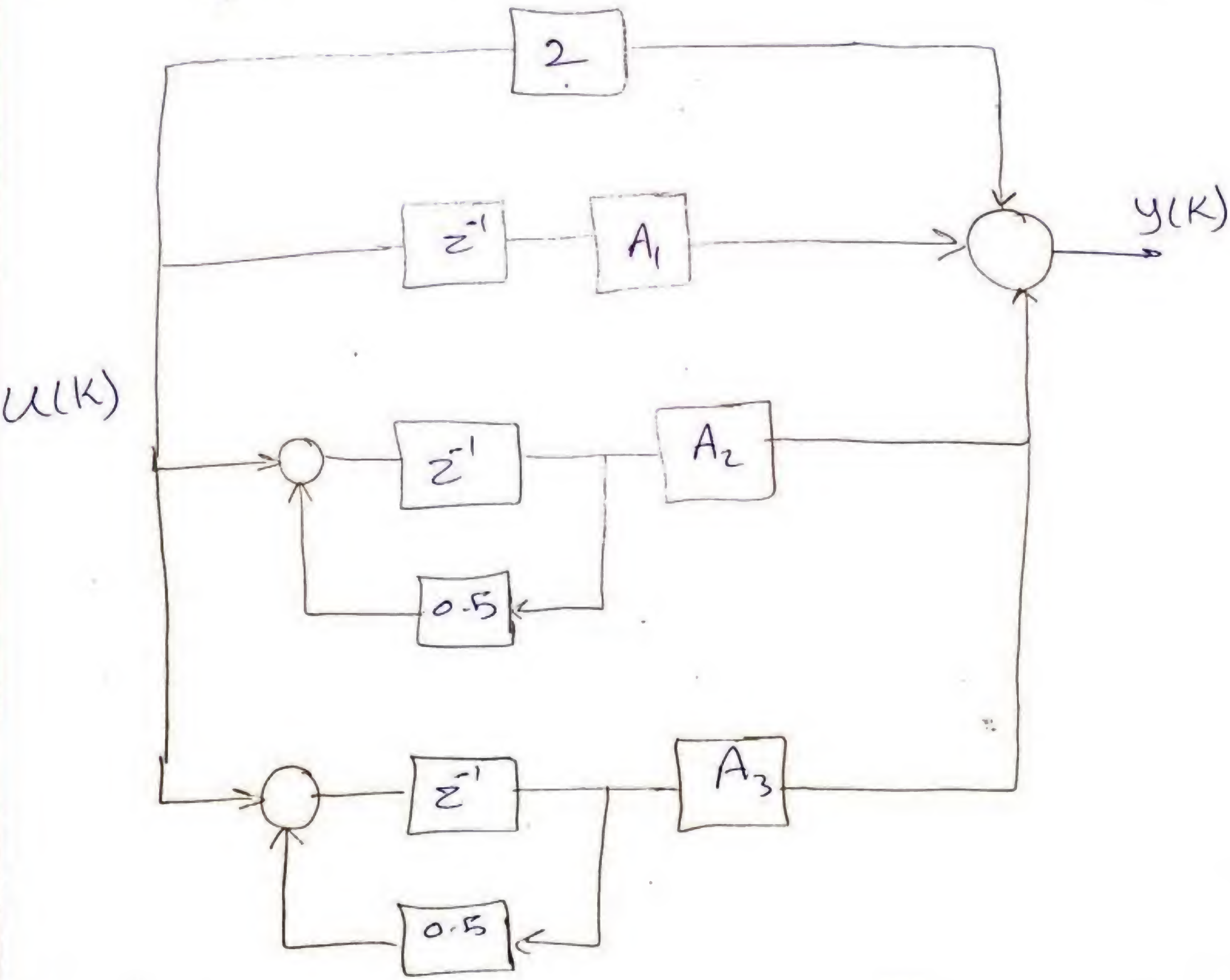
using P.F

$$= 2 + \frac{A_1}{z} + \frac{A_2}{z+0.5} + \frac{A_3}{z-0.5}$$

$$T.F = 2 - \frac{4}{z} \quad \frac{\quad}{z+0.5} \quad \frac{\quad}{z-0.5}$$

$$x(k+1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} x(k) + 2\, u(k)$$



state-space Analysis

given
$$\boxed{\begin{aligned} x(K+1) &= A\,x(K) + B\,u(K) \\ y(K) &= C\,x(K) + D\,u(K) \end{aligned}}$$

$A, B, C, D \longrightarrow$ given

1 T.F

$$x(K+1) = A\,x(K) + B\,u(K) \longrightarrow z\,x(z)$$

$$z\,x(z) = A\,x(z) + B\,u(z)$$

$$z\,x(z) - A\,x(z) = B\,u(z)$$

$$(zI - A)\,x(z) = B\,u(z)$$

$$\boxed{x(z) = B\,(zI - A)^{-1}\,u(z)} \longrightarrow 1$$

$$y(K) = C\,x(K) + D\,u(K)$$

$$y(z) = C\,x(z) + D\,u(z) \longrightarrow 2$$

from ① in ②

$$Y(z) = C(zI-A)^{-1} B\,U(z) + D\,U(z)$$

$$= \left(C(zI-A)^{-1} B + D\right) U(z)$$

$$\boxed{T.F = \frac{Y(z)}{U(z)} = C(zI-A)^{-1} B + D}$$

[2] ch. equation

$$|zI-A| = 0$$

$T.F = \frac{\text{بسط}}{\text{مقام}}$

له مقام = صفر

[3] check the system Controllability & observability :-

a) Controllability

→ The system is controllable • if for any change of an external input, produce change in internal states of the system.

observability:

The system is observable, if we can determine or estimate the states' values from the relation between input and output or by the history information for the o/p and i/p.

→ controllability matrix (M_c):

$$M_c = (B \quad AB \quad A^2B \ldots\ldots A^{n-1}B)$$ $n \rightarrow$ system order.

2nd order $M_c = (B \quad AB)$

3rd order $M_c = (B \quad AB \quad A^2B)$

if $|M_c| \neq 0 \rightarrow$ system is Controllable.

* observability matrix (M_o)

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} ; \quad \text{if } |M_o| \neq 0$$

↳ system is observable.

2nd order

$$M_o = \begin{pmatrix} C \\ CA \end{pmatrix}$$

3rd order

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$$

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4) system response

Let $\phi(z) = (zI - A)^{-1}$

$$x(z) = \phi(z) \cdot z \, x(0) + \phi(z) \, B \, u(z)$$

$\downarrow z^{-1} \cdot T$

$$x(K) = \dots$$

System response : $y(K) = C \, x(K) + D u(K)$

$\boxed{Ex}$

$$X(K+1) = \begin{pmatrix} 0 & 0.6321 \\ -1 & 1 \end{pmatrix} X(K) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(K)$$

$$y(K) = \begin{pmatrix} 1 & -0.6321 \end{pmatrix} X(K)$$

① find T.F

② check stability

$$\underline{\underline{T.F}} = C(ZI-A)^{-1}B + \cancel{D} \to 0$$

$$(ZI-A) = \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} - \begin{pmatrix} 0 & 0.6321 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} Z & -0.6321 \\ 1 & Z-1 \end{pmatrix}$$

$$(ZI-A)^{-1} = \frac{1}{Z(Z-1)+0.6321} \begin{pmatrix} Z-1 & 0.6321 \\ -1 & Z \end{pmatrix}$$

$$T.F = \begin{pmatrix} 1 & -0.6321 \end{pmatrix} \frac{1}{(Z-1)Z+0.6321} \begin{pmatrix} Z-1 & 0.6321 \\ -1 & Z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{Z(Z-1)+0.6321} \begin{pmatrix} 1 & -0.6321 \end{pmatrix} \begin{pmatrix} Z-1+0.6321 \\ -1+Z \end{pmatrix}$$

$$T.F = \frac{z - 1 + 0.6321 + 0.6321 - 0.6321 z}{z^2 - z + 0.6321}$$

$$\boxed{T.F = \frac{0.3679 z + 0.2642}{z^2 - z + 0.6321}}$$

ch. eqn $z^2 - z + 0.6321 = 0$

$Z_{1,2} = 0.5 \pm J\, 0.618$

poles

$|Z_{1,2}| = 0.795 < 1 \longrightarrow$ system stable

Ex

$$x(K+1) = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix} x(K) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(K)$$

$$y(K) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(K)$$

if $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; find unit step response.

$$X(z) = \phi(z) z\, x(0) + \phi(z) B u(z)$$

$$\phi(z) = (zI - A)^{-1}$$

$$zI - A = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix} = \begin{pmatrix} z-0.8 & 0 \\ 0 & z-0.2 \end{pmatrix}$$

$$(zI - A)^{-1} = \frac{1}{(z-0.8)(z-0.2)} \begin{pmatrix} z-0.2 & 0 \\ 0 & z-0.8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix} = (zI - A)^{-1} = \phi(z)$$

$$X(z) = \begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix} * z * \begin{pmatrix} 1 \\ 1 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix} * \begin{pmatrix} 1 \\ 1 \end{pmatrix} * \left(\underbrace{\frac{z}{z-1}}_{\text{unit step}}\right)$$

$$X(z) = \begin{bmatrix} \dfrac{z}{z-0.8} \\[2ex] \dfrac{z}{z-0.2} \end{bmatrix} + \begin{bmatrix} \dfrac{1}{z-0.8} \\[2ex] \dfrac{1}{z-0.2} \end{bmatrix} \dfrac{z}{z-1}$$

$$= \begin{bmatrix} \dfrac{z}{z-0.8} \\[2ex] \dfrac{z}{z-0.2} \end{bmatrix} + \begin{bmatrix} \dfrac{z}{(z-0.8)(z-0.2)} \\[2ex] \dfrac{z}{(z-0.2)(z-1)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{z}{z-0.8} \\[2ex] \dfrac{z}{z-0.2} \end{bmatrix} + \begin{bmatrix} \dfrac{-5z}{z-0.8} + \dfrac{5z}{z-1} \\[2ex] \dfrac{-1.25z}{z-0.2} + \dfrac{1.25z}{z-1} \end{bmatrix}$$

مع احنا خرجنا ال Z برة لحد ما نحسب ال ~~Z~~ Partial fraction

$\underline{z^{-1}.T}$

$$X(k) = \begin{bmatrix} (0.8)^k \\[2ex] (0.2)^k \end{bmatrix} + \begin{pmatrix} -5(0.8)^k + 5 \\[2ex] -1.25(0.2)^k + 1.25 \end{pmatrix}$$

$$X(K) = \begin{pmatrix} (0.8)^K - 5(0.8)^K + 5 \\ (0.2)^K - 1.25(0.2)^K + 1.25 \end{pmatrix}$$

$$= \begin{pmatrix} -4(0.8)^K + 5 \\ -0.25(0.2)^K + 1.25 \end{pmatrix}$$

$\rightarrow$ unit step response

$$y(K) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} -4(0.8)^K + 5 \\ -0.25(0.2)^K + 1.25 \end{pmatrix}$$

$$= -4(0.8)^K + 5 - 0.25(0.2)^K + 1.25$$

$$\boxed{y(K) = -4(0.8)^K - 0.25(0.2)^K + 6.25}$$

$\boxed{Ex}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} ; \quad B = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} ; \quad C = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$$

check system Controllability and observability

$$\mathcal{M}_c = \begin{pmatrix} B & AB & A^2 B \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$$

$$A^2 B = A \cdot AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -20 \end{pmatrix}$$

$$\mathcal{M}_c = \begin{bmatrix} {}^{+}0 & 3 & -2 \\ {}^{-}1 & 11 & 7 \\ {}^{+}3 & -5 & -20 \end{bmatrix}$$

$$|M_c| = -1 \begin{vmatrix} 3 & -2 \\ -5 & -20 \end{vmatrix} + 3 \begin{vmatrix} 3 & -2 \\ 11 & 7 \end{vmatrix}$$

$$= 70 + 129 = 199 \neq 0 \rightarrow \text{system controllable}$$

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$$

$$CA = (1 \quad 0 \quad 2) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} = (-1 \quad -4 \quad -1)$$

$$CA^2 = CA \cdot A = (-1 \quad -4 \quad -1) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix}$$

$$= (0 \quad -6 \quad -12)$$

$$M_o = \begin{pmatrix} \overset{+}{1} & 0 & 2 \\ \overset{-}{-1} & -4 & -1 \\ \overset{+}{0} & -6 & -12 \end{pmatrix}$$

$$|M_o| = 1 \begin{vmatrix} -4 & -1 \\ -6 & -12 \end{vmatrix} + 1 \begin{vmatrix} 0 & 2 \\ -6 & -12 \end{vmatrix} = 54 \neq \text{system observable}$$